Interval: Indonesian Journal of Mathematical Education
Vol. 3, No. 1, June 2025, pp. 69~81
ISSN: 3021-7857, DOI: 10.37251/ijome.v3i1.1588

Modification of the Fourth Order Runge Kutta Method Based on the Contra Harmonic Average

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Article Info

Article history:

Received Mar 3, 2025
Revised Apr 19, 2025
Accepted May 4, 2025
Online First May 5, 2025

Keywords:

Contra Harmonics
Error
Literature review
RKKCM
Runge-Kutta 4 Kutta

ABSTRACT

Purpose of the study: Aims to modify the fourth order runge-kutta method based on the contra harmonic mean. This study discusses the theoretical modification of the fourth order runge-kutta method.

Methodology: The research was conducted using a literature study method. The writing begins by introducing the general Runge-Kutta method up to the nth order. Then this general form will be specialized up to the fourth order. In addition, the contra harmonic average will also be introduced. After obtaining the general form of the fourth order Runge-Kutta and the contra harmonic average, the last step is to modify the two general forms so that a new formula will be obtained.

Main Findings: Based on the results of the study, it was found that the fourth-order Kutta Runge-Kutta modification method has the following equation form: $y_{i+1} = y_i + \frac{h}{4}\left(\frac{k_1^2+k_2^2}{k_1+k_2} + 2\frac{k_2^2+k_3^2}{k_2+k_3} + \frac{k_3^2+k_4^2}{k_3+k_4}\right)$ with an error of order $5\left(O(h^5)\right)$. The results of numerical simulations show that the modified fourth-order Runge-Kutta Kutta is better than the modified fourth-order Runge-Kutta Kutta.

Novelty/Originality of this study: Numerical simulation using the RKKCM method shows that the results of this method have better accuracy compared to the fourth-order Runge-Kutta method before modification.

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1. INTRODUCTION

Problems involving mathematical models often arise in various scientific disciplines such as in Physics, Chemistry, Economics, or in engineering problems given in the form of first-order or second-order differential equations [1]-[6]. Often these mathematical models appear in non-ideal or complicated forms [7]-[9]. These complicated mathematical models sometimes cannot be solved by standard analytical methods to obtain the exact solution [10]-[12]. If the analytical method can no longer be applied, then the actual solution to the problem can still be found using numerical calculations used to formulate mathematical problems so that they can be solved with ordinary arithmetic or calculation operations.

Several one-step numerical calculations that are often used to solve ordinary differential equations (ODEs) such as Euler, Heun, Taylor, and Runge-Kutta are some examples of methods that are classified as numerical methods. The Runge-Kutta method is known as a method that has better accuracy compared to the three methods.

The Runge-Kutta method is another alternative to the Taylor method that does not require derivative calculations [13]-[15]. This method attempts to obtain a high degree of accuracy and at the same time avoids the need to find higher derivatives by evaluating the function *f(x,y)* at selected points in each interval.

In general, the form of the Runge-Kutta method is written as follows:

$$y_{i+1} = y_i + a_1 k_1 + a_2 k_2 + \cdots + a_n k_n \qquad \ldots(1)$$

while the fourth order Runge-Kutta method is:

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad \ldots(2)$$

Several modifications have been made to the fourth-order Runge-Kutta method. Based on its general form, the fourth-order Runge-Kutta method can be modified based on the Arithmetic mean as follows:

$$y_{i+1} = y_i + \frac{h}{3}\left(\frac{k_1+k_2}{2} + \frac{k_2+k_3}{2} + \frac{k_3+k_4}{2}\right) \qquad \ldots(3)$$

In addition to using the Arithmetic Mean, a modification of the 4th Order Runge-Kutta method based on the Geometric Mean has been carried out by Evans [16], and obtained:

$$y_{i+1} = y_i + \frac{h}{3}\left(\sqrt{k_1 k_2} + \sqrt{k_2 k_3} + \sqrt{k_3 k_4}\right) \qquad \ldots(4)$$

with $k_1 = f(x_i, y_i)$

$$k_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_1\right)$$

$$k_3 = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{16}(-k_1 + 9k_2)\right)$$

$$k_4 = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{24}(-3k_1 + 5k_2 + 22k_3)\right)$$

Furthermore, Sanugi and Evans [17] carried out a fourth-order Runge-Kutta modification based on the Harmonic mean,

$$y_{i+1} = y_i + \frac{2h}{3}\left(\frac{k_1 k_2}{k_1+k_2} + \frac{k_2 k_3}{k_2+k_3} + \frac{k_3 k_4}{k_3+k_4}\right) \qquad \ldots(5)$$

with $k_1 = f(x_i, y_i)$

$$k_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_1\right)$$

$$k_3 = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{8}(-k_1 + 5k_2)\right)$$

$$k_4 = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{8}(-5k_1 + 7k_2 + 18k_3)\right)$$

Ababneh and Rozita [18] introduced a modification of the third-order Runge Kutta method based on the average Contra Harmonic for differential equation problems, which is given by:

$$y_{i+1} = y_i + h\left[\frac{1}{4}\frac{k_1^2+k_2^2}{k_1+k_2} + \frac{3}{4}\frac{k_2^2+k_3^2}{k_2+k_3}\right] \qquad \ldots(6)$$

with $k_1 = f(x_i, y_i)$

$$k_2 = f\left(x_i + h\frac{2}{3}, y_n + h\frac{2}{3}k_1\right)$$

$$k_3 = f\left(x_i + h\frac{4}{21}\left(3+\sqrt{2}\right), y_i + h\frac{4}{21}\left(3+\sqrt{2}\right)k_2\right)$$

$k_4 = f(x_i + \frac{h}{2}, y_i + \frac{h}{8}(-5k_1 + 7k_2 + 18k_3))$

The Runge-Kutta method is one of the most widely used numerical methods in solving initial value problems in ordinary differential equations (ODEs) [19]-[21]. Among the various variants of this method, the fourth-order Runge-Kutta method (RK4) is widely known for its balance between accuracy and computational efficiency [22]-[24]. Various studies have been conducted to modify and improve the RK4 method, both in terms of mathematical approach, calculation efficiency, and numerical stability in dealing with stiff problems. However, most of the modifications that have been carried out still use the arithmetic or harmonic mean approach in formulating the stage coefficients. Studies on the use of other forms of averages, such as the contra harmonic mean, are still relatively limited. In fact, the contra harmonic mean has unique characteristics that can contribute to increasing the accuracy of estimating numerical solution values.

Previous studies, such as those conducted by Evans [16] and Sanugi & Evans [17], have proposed modifications to the Runge-Kutta method with a harmonic mean approach, but have not examined the potential application of the contra harmonic mean as a basis for modification. Meanwhile, Ababneh and Rozita [18] have explored the application of the contra harmonic mean in third-order Runge-Kutta, but no study has specifically modified the fourth-order Runge-Kutta method with this approach. This is the research gap that this study aims to answer.

With the increasing complexity of mathematical models in various fields of science and engineering, numerical methods are needed that can provide more accurate results without increasing significant computational burden [25]-[27]. Therefore, it is important to explore new approaches in modifying numerical methods, especially the fourth-order Runge-Kutta method, to improve the accuracy and efficiency in solving ODEs.

This study offers a novelty by modifying the fourth-order Runge-Kutta method using the contra harmonic mean as the basis for calculating the stages. The results of numerical simulations show that this approach produces better accuracy than the conventional RK4 method. This is a new contribution to the development of numerical methods based on Runge-Kutta. The main objective of this study is to modify the fourth-order Runge-Kutta method using the contra harmonic mean, and to evaluate the performance of the modified method through numerical simulations on initial value problems. This study also aims to prove that the proposed modification provides increased accuracy compared to the classical RK4 method.

Based on the research that has been carried out to modify the Runge-Kutta Method, the author is interested in developing the modification made by Ababneh and Rozita [18] by applying the average Contra Harmonic to the Runge-Kutta method of order 4 Kutta. Therefore, the formulation in this research is how to modify the Kutta Fourth Order Runge-Kutta Method based on the average Contra Harmonic.

2. RESEARCH METHOD

This study discusses theoretically the modification of the Runge-Kutta method of order 4 Kutta. Therefore, the study was conducted using a literature study method [28]-[30] that is useful for collecting data and information needed from books, journals, and sources from the internet.

The writing begins by introducing the Runge-Kutta Method in general up to the nth order. Then this general form will be specialized up to Order 4. In addition, the Contra Harmonic average will also be introduced. After obtaining the general form of Runge-Kutta Order 4 and the Contra Harmonic average, the last step is to modify the two general forms so that a new formula will be obtained. Described in the Flowchart in Figure 1.

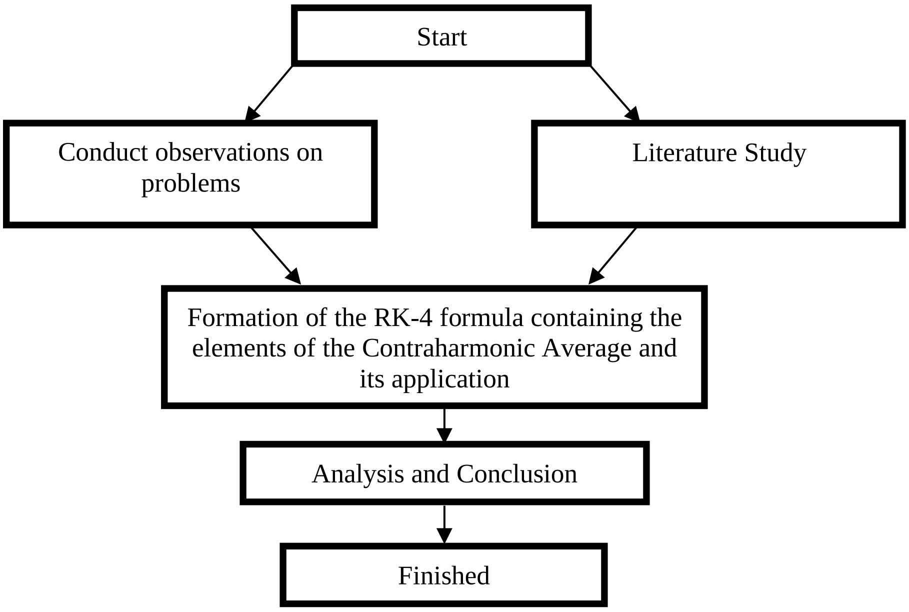

Figure 1. Flowchart for Compiling Final Assignments

3. RESULTS AND DISCUSSION

3.1. Modification of RK-4 Kutta Method Using Counter Harmonic Average

Several modifications of the Runge-Kutta Method have been produced, such as modifications based on Arithmetic, Geometric, Harmonic, and even Counterharmonic averages as introduced by Ababneh to Runge-Kutta Order 3 [31]-[33]. In this research, the author will prove the Modification of the Runge-Kutta Method Order 4 Kutta based on Counterharmonic Average. Take another look at the general form of Runge-Kutta Order 4 Kutta:

$$y_{i+1} = y_i + \frac{h}{8}(k_1 + 3k_2 + 3k_3 + k_4) \quad \ldots(7)$$

Based on equation (7) a new formula can be formed which contains arithmetic elements as follows:

$$y_{i+1} = y_i + \frac{h}{4}\left(\frac{k_1}{2} + \frac{3k_2}{2} + \frac{3k_3}{2} + \frac{k_4}{2}\right)$$

or

$$y_{i+1} = y_i + \frac{h}{4}\left(\frac{k_1+k_2}{2} + \frac{k_2+k_3}{2} + \frac{k_2+k_3}{2} + \frac{k_3+k_4}{2}\right) \quad \ldots(8)$$

Equation (8) is known as the 4th Order Runge-Kutta based on the Arithmetic Mean. where the form $\frac{k_i+k_{i+1}}{2}$ is the Arithmetic mean for two variables. If the Arithmetic mean is squared and divided by the Arithmetic mean itself, the 4th Order Runge-Kutta equation will be formed as follows:

$$y_{i+1} = y_i + \frac{h}{4}\left(\frac{\frac{k_1^2+k_2^2}{2} + \frac{k_2^2+k_3^2}{2} + \frac{k_2^2+k_3^2}{2} + \frac{k_3^2+k_4^2}{2}}{\frac{k_1+k_2}{2} + \frac{k_2+k_3}{2} + \frac{k_3+k_4}{2}}\right)$$

can be simplified to

$$y_{i+1} = y_i + \frac{h}{4}\left(\frac{k_1^2+k_2^2}{k_1+k_2} + 2\frac{k_2^2+k_3^2}{k_2+k_3} + \frac{k_3^2+k_4^2}{k_3+k_4}\right) \quad \ldots(9)$$

with

$$k_1 = f(y_1) = hf$$

$k_2 = f(y_1 + hq_{21}k_1)$

$k_3 = f(y_1 + hq_{31}k_1 + hq_{32}k_2)$

$k_4 = f(y_1 + hq_{41}k_1 + hq_{42}k_2 + hq_{43}k_3)$ ...(10)

The form $\frac{k_i^2+k_{i+1}^2}{2}$ is defined as the Counter Harmonic mean, so equation (9) is known as the modified Kutta 4th Order Runge-Kutta method based on the Counter Harmonic mean.

Describe k_1, k_2, k_3, and k_4 and into the form of a first-order Taylor series for the function of two variables (11) so that equations (12), (13), (14), and (15) will be obtained.

$$f(x,y) = \sum_{i=0}^{\infty} \frac{1}{i!}\left[(x - x_n)\frac{\partial}{\partial x} + (y - y_n)\frac{\partial}{\partial y}\right]^i f(x_n, y_n) \quad ...(11)$$

$k_1 = f_i$ ...(12)

$k_2 = f(y_i + q_{21}k_1)$

$$k_2 = f + hq_{21}ff_y + \frac{h^2}{2}q_{21}^2 f^2 f_{yy} + \frac{h^3}{6}q_{21}^3 f^3 f_{yyy} + \cdots \quad ...(13)$$

$k_3 = f(y_i + q_{31}k_1 + q_{32}k_2)$

$$k_3 = f + hAff_y + h^2\left(q_{21}q_{32}ff_y^2 + \frac{1}{2}A^2 f^2 f_{yy}\right) +$$

$$h^3\left(\frac{1}{2}q_{21}^2 q_{32} f^2 f_y f_{yy} + q_{21}Aq_{32}f^2 f_y f_{yy} + \frac{1}{6}A^3 f^3 f_{yyy}\right) + \cdots \quad ...(14)$$

$$k_4 = f + hBff_y + h^2\left(q_{21}q_{42}ff_y^2 + Aq_{43}ff_y^2 + \frac{1}{2}B^2 f^2 f_{yy}\right) +$$

$$h^3\left(\frac{1}{2}q_{21}^2 q_{42}Bf^2 f_y f_{yy} + q_{43}\frac{A^2}{2}f^2 f_y f_{yy} + B(q_{21}q_{42} + q_{43}A)f^2 f_y f_{yy} + q_{43}q_{32}q_{21}ff_y^3 + \frac{1}{6}B^3 f^3 f_{yyy}\right)$$

$$+\cdots \quad ...(15)$$

In general, equations (12) - (15) will be substituted to obtain the function y_{i+1} with parameters q_{21}, q_{31}, q_{32}, q_{41}, q_{42}, and q_{43}. However, because there is a division of two series (Contra Harmonic average), direct substitution cannot be done. Then cross multiplication will be carried out so that this problem will be reduced and there is a form $(k_1 + k_2)(k_2 + k_3)(k_3 + k_4)$, which can be written as follows:

$$y_{i+1} = y_i + \frac{numerator}{denominator} \quad ...(16)$$

Next, compare equation (16) with the Taylor series, so that we obtain:

$$y_i + \frac{numerator}{denominator} = y_i + T$$

or

$$numerator = T \times denominator \quad ...(17)$$

with

$$numerator = \frac{h}{4}(k_1^2 + k_2^2)(k_2 + k_3)(k_3 + k_4) + 2(k_2^2 + k_3^2)(k_1 + k_2)(k_3 + k_4) + (k_3^2 + k_4^2)(k_1 + k_2)(k_3 + k_4)$$

$$denominator = (k_1 + k_2)(k_2 + k_3)(k_3 + k_4)$$

$$T = hf + \frac{h^2}{2}ff_y + \frac{h^3}{6}\left(ff_y^2 + f^2 f_y\right) + \frac{h^4}{24}\left(f^3 f_{yyy} + 4f^2 f_y f_{yy} + ff_y^3\right)$$

Then substitute the values of k_1, k_2, k_3, and k_4 obtained in equations (12), (13), (14), and (15) into equation (17) and obtain:

$h^2f^4f_y$: $3q_{21} + 3A + B = 4$

$h^3f^5f_{yy}$: $\frac{3}{2}q_{21}^2 + \frac{3}{2}A^2 + \frac{1}{2}B^2 = \frac{4}{3}$

$h^3f^4f_y^2$: $\frac{9}{2}q_{21}^2 + \frac{9}{2}A^2 + B^2 + \frac{3}{2}AB + \frac{5}{2}q_{21}B + 4q_{21}A + 3q_{32}q_{21} + q_{42}q_{21} + q_{43}A - 4A - 2B = \frac{4}{3}$

$h^4f^6f_{yyy}$: $\frac{1}{2}q_{21}^3 + \frac{1}{2}A^3 + \frac{1}{6}B^3 = \frac{1}{3}$

$h^4f^5f_yf_{yy}$: $\frac{9}{2}q_{21}^3 + \frac{9}{2}A^3 + B^3 + 2q_{21}^2A + 2q_{21}A^2 + \frac{5}{4}q_{21}B^2 + \frac{5}{2}q_{21}^2B + \frac{1}{2}q_{21}^2q_{42} + \frac{1}{2}q_{43}B^2$

$+B(q_{42}q_{21} + q_{43}A) + \frac{3}{4}A^2B + \frac{3}{4}AB^2 + \frac{3}{2}q_{21}^2q_{32} + 3q_{21}q_{32}A - 2q_{21}^2 - 2A^2 - B^2$

$-\frac{4}{3}q_{21} - \frac{4}{3}A - \frac{2}{3}B = \frac{4}{3}$

$h^4f^4f_y^3$: $3q_{21}^2 + 3q_{21}A^2 + \frac{5}{2}q_{21}^2B + q_{21}B^2 + 4q_{21}^2q_{32} + \frac{5}{2}q_{21}^2q_{42} + \frac{3}{2}q_{21}q_{32} + \frac{3}{2}q_{21}q_{42} + \frac{5}{2}q_{21}q_{43}A$

$+9q_{21}q_{21}A + 2q_{43}AB + q_{21}AB + q_{21}q_{32}q_{43} + 2q_{21}q_{43}B + \frac{3}{2}q_{43}A^2 + A^2B + \frac{1}{2}AB^2 + \frac{3}{2}q_{21}^3$

$+\frac{3}{2}A^3 - 4q_{21}q_{32} - 2q_{21}q_{42} - 2q_{43}A - 3q_{21}A - AB - 2q_{21}B - \frac{4}{3}q_{21} - \frac{4}{3}A - \frac{2}{3}B - q_{21}^2$

$-A^2 = \frac{1}{3}$ ...(18)

To make it easier to get the parameter values, a simplification is first carried out by taking the values:

$A = \frac{2}{3}$ and $B = 1$

then the following equation is obtained:

$h^2f^4f_y$: $3q_{21} = 1$

$h^3f^5f_{yy}$: $\frac{3}{2}q_{21}^2 = \frac{1}{6}$

$h^3f^4f_y^2$: $\frac{7}{6}q_{21} - \frac{9}{2}q_{21}^2 + 3q_{32}q_{21} + q_{42}q_{21} + \frac{2}{3}q_{43} = \frac{2}{3}$

$h^4f^6f_{yyy}$: $\frac{1}{2}q_{21}^3 = \frac{1}{54}$

$h^4f^5f_yf_{yy}$: $\frac{9}{2}q_{21}^3 + \frac{7}{12}q_{21}^2 + \frac{29}{36}q_{21} + \frac{3}{2}q_{21}^2q_{32} + 2q_{21}q_{32} + \frac{1}{2}q_{21}^2q_{42} + q_{21}q_{42} + \frac{7}{6}q_{43}\frac{29}{18}$

$h^4f^4f_y^3$: $\frac{3}{2}q_{21}^3 + \frac{7}{2}q_{21}^2 - \frac{7}{3}q_{21} + 4q_{21}^2q_{32} + \frac{7}{2}q_{21}q_{32} + \frac{5}{2}q_{21}^2q_{42} + q_{21}q_{42} + \frac{5}{3}q_{21}q_{42} + \frac{2}{3}q_{43}$

$+q_{21}q_{32}q_{43} = \frac{16}{9}$ ...(19)

Then substitute the value q_{21} = 1/3 into equation (19) to obtain the following new equation:

$q_{32} + \frac{1}{3}q_{42} + \frac{2}{3}q_{43} = \frac{10}{9}$

$\frac{5}{6}q_{32} + \frac{7}{18}q_{42} + \frac{7}{6}q_{43} = \frac{10}{9}$

$\frac{29}{18}q_{32} + \frac{11}{18}q_{42} + \frac{11}{9}q_{43}\frac{1}{3}q_{32}q_{43} = \frac{19}{9}$ ...(20)

by solving the system of equations in (20), the values are obtained $q_{21} = \frac{1}{3}$, $q_{31} = \frac{5}{18} - \frac{\sqrt{73}}{18}$, $q_{32} = \frac{7}{18} + \frac{\sqrt{73}}{18}$, $q_{41} = \frac{-5}{3} + \frac{\sqrt{73}}{3}$, $q_{42} = \frac{19}{6} - \frac{\sqrt{73}}{2}$ and $q_{43} = \frac{-1}{2} + \frac{\sqrt{73}}{6}$.

The final step is to substitute all the parameter values that have been obtained into equation (10) and obtain the following equation form:

$k_1 = hf(y_i) = hf$

$k_2 = hf\left(y_i + \frac{1}{3}k_1\right)$

$k_3 = hf\left(y_i + \left(\frac{5}{18} + \frac{\sqrt{73}}{18}\right)k_1 - \left(\frac{7}{18} + \frac{\sqrt{73}}{18}\right)k_2\right)$

$k_4 = hf\left(y_i + \left(\frac{-5}{3} + \frac{\sqrt{73}}{3}\right)k_1 - \left(\frac{19}{6} - \frac{\sqrt{73}}{2}\right)k_2 - \left(\frac{1}{2} - \frac{\sqrt{73}}{6}\right)k_3\right)$ ...(21)

Equations (9) and (21) are known as the Fourth Order Runge-Kutta Method based on Contra Harmonic Mean.

Example 1. Determine the approximation of the following differential equation:

$y' = y$ with initial conditions $y(0) = 1$ and $h = 0.1$

And for comparison an exact solution is given $y = e^x$

Solution:
The valuesk_1, k_2, k_3, dan k_4 will be determined first as follows:

$k_1 = hf(0) = 0.1$

$k_2 = y(0) + \frac{1}{3}hk_1$

$k_2 = 1 + \frac{0.1}{3}(1)$

$k_2 = 1.00333$

$k_3 = y(0) + \left(\frac{5}{18} - \frac{\sqrt{73}}{18}\right)hk_1 + \left(\frac{7}{18} + \frac{\sqrt{73}}{18}\right)hk_2$

$k_3 = 1 + 0.086472$

$k_3 = 1.08647$

$k_4 = y(0) + \left(\frac{-5}{3} + \frac{\sqrt{73}}{3}\right)hk_1 + \left(\frac{19}{6} - \frac{\sqrt{73}}{2}\right)hk_2 + \left(\frac{-1}{2} + \frac{\sqrt{73}}{6}\right)hk_3$

$k_4 = 1 + 0.00114$

$k_4 = 1.00114$

Next, substitute the obtained values into the fourth-order Runge-Kutta general equation based on the Contraharmonic average, and we get:

$y_{i+1} = y_i + \frac{h}{4}\left(\frac{k_1^2+k_2^2}{k_1+k_2} + 2\frac{k_2^2+k_3^2}{k_2+k_3} + \frac{k_3^2+k_4^2}{k_3+k_4}\right)$

or

$y_1 = 1.10143$

Meanwhile, the exact solution is $y = e^{0.1} = 1.10517$ so that by comparing the exact solution and the approximate solution (RKCM Kutta) the error value will be $E = 0.00374$.

3.2. Errors in the Fourth Order Runge-Kutta Method Kutta based on the Contra Harmonic Average

To obtain the error in the Fourth Order Runge-Kutta Method Kutta based on the Counter Harmonic Mean, the steps are the same as determining the previous RKCM formula derivation. By substituting the obtained parameter values into equation (17) and expanding it to Order 5 (h^5), the error for the RKCM method will be obtained as follows:

$$galat_{RKKCM} = h^5\left(\frac{17}{243}f^6 f_y f_{yyy} + \frac{11}{162}f^6 f_{yy}^2 + \left(\frac{1}{3} - \frac{2\sqrt{73}}{81}\right)f^5 f_y^2 f_{yy} + \left(\frac{73}{243} - \frac{5\sqrt{73}}{243}\right)f^4 f_y^4\right) \quad \ldots(22)$$

3.3. Numerical Simulation

The following is the solution of $y' = y$ with initial conditions $y(0) = 1$ and $h = 0.01$ for the exact solution of $y = e^x$ on the interval [0,1] using the modified RK-4 Kutta method.

- RK-4 kutta modification method
  - for $h = 0.01$

$k_1 = hy(0) = 0.01$

$k_2 = y(0) + \frac{1}{3}hk_1$

$k_2 = 1 + \frac{0.01}{3}(1)$

$k_2 = 1.00003$

$k_3 = y(0) + \left(\frac{5}{18} - \frac{\sqrt{73}}{18}\right)hk_1 + \left(\frac{7}{18} + \frac{\sqrt{73}}{18}\right)hk_2$

$k_3 = 1 + 0.086472$

$k_3 = 1.08647$

$k_4 = y(0) + \left(\frac{-5}{3} + \frac{\sqrt{73}}{3}\right)hk_1 + \left(\frac{19}{6} - \frac{\sqrt{73}}{2}\right)hk_2 + \left(\frac{-1}{2} + \frac{\sqrt{73}}{6}\right)hk_3$

$k_4 = 1 + 0.00114$

$k_4 = 1.00114$

$y(1) = y(0) + \frac{h}{4}\left(\frac{k_1^2 + k_2^2}{k_1 + k_2} + 2\frac{k_2^2 + k_3^2}{k_2 + k_3} + \frac{k_3^2 + k_4^2}{k_3 + k_4}\right)$

$y(1) = 1.010006$

- for $h = 0.02$

$k_1 = hy(0) = 0.02$

$k_2 = y(0) + \frac{1}{3}hk_1$

$k_2 = 1 + \frac{0.02}{3}(1)$

$k_2 = 1.00013$

$k_3 = y(0) + \left(\frac{5}{18} - \frac{\sqrt{73}}{18}\right) hk_1 + \left(\frac{7}{18} + \frac{\sqrt{73}}{18}\right) hk_2$

$k_3 = 1 + 0.01719$

$k_3 = 1.01719$

$k_4 = y(0) + \left(\frac{-5}{3} + \frac{\sqrt{73}}{3}\right) hk_1 + \left(\frac{19}{6} - \frac{\sqrt{73}}{2}\right) hk_2 + \left(\frac{-1}{2} + \frac{\sqrt{73}}{6}\right) hk_3$

$k_4 = 1 - 0.00283$

$k_4 = 0.99716$

$y(1) = y(0) + \frac{h}{4}\left(\frac{k_1^2+k_2^2}{k_1+k_2} + 2\frac{k_2^2+k_3^2}{k_2+k_3} + \frac{k_3^2+k_4^2}{k_3+k_4}\right)$

$y(1) = 1.020028$

- for $h = 0.03$

$k_1 = hy(0) = 0.03$

$k_2 = y(0) + \frac{1}{3}hk_1$

$k_2 = 1 + \frac{0.03}{3}(1)$

$k_2 = 1.00030$

$k_3 = y(0) + \left(\frac{5}{18} - \frac{\sqrt{73}}{18}\right) hk_1 + \left(\frac{7}{18} + \frac{\sqrt{73}}{18}\right) hk_2$

$k_3 = 1 + 0.02573$

$k_3 = 1.02573$

$k_4 = y(0) + \left(\frac{-5}{3} + \frac{\sqrt{73}}{3}\right) hk_1 + \left(\frac{19}{6} - \frac{\sqrt{73}}{2}\right) hk_2 + \left(\frac{-1}{2} + \frac{\sqrt{73}}{6}\right) hk_3$

$k_4 = 1 - 0.00367$

$k_4 = 0.99632$

$y(1) = y(0) + \frac{h}{4}\left(\frac{k_1^2+k_2^2}{k_1+k_2} + 2\frac{k_2^2+k_3^2}{k_2+k_3} + \frac{k_3^2+k_4^2}{k_3+k_4}\right)$

$y(1) = 1.030072$

Do the iteration process above continuously until the interval [0.1] with $h = 0.01$. Then the iteration results can be shown in the plotting graph in Figure 2.

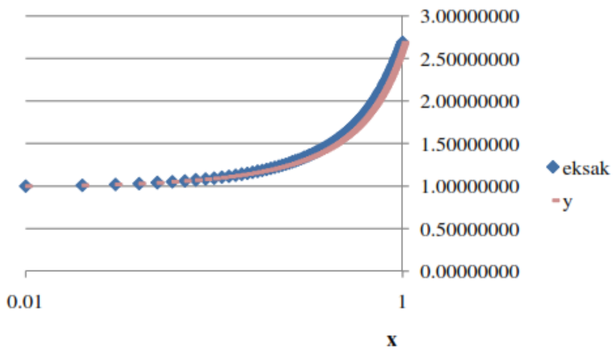

Figure 2. Plotting graph for the case $y' = y$ in the interval [0,1] and $h = 0.01$

Based on the results of the numerical calculations and by looking at the plotting graph of the calculation results, it is clear that the approximate value using the RKKCM modification method is more significant when compared to the RK-4 Kutta method before it was modified.

Modification of the fourth-order Runge-Kutta (RK4) method with the contra harmonic average approach provides a significant contribution in improving the accuracy of numerical solutions to initial value problems. This new formulation considers the nature of the contra harmonic mean, which gives greater weight to larger values in the data. This approach is mathematically more sensitive to large changes in the solution derivatives (slopes), so it can capture sharper and more complex system dynamics compared to the usual arithmetic mean [34]-[36].

Numerical simulation results show that the modified method called RKKCM (Runge-Kutta Kutta Contra Harmonic Mean) is able to produce solution values that are closer to the exact solution than the classical RK4 method. This shows that the use of the contra harmonic mean in compiling the RK stages provides advantages in reducing local errors and increasing the stability of the method. In the context of numerical implementation, this new formula remains efficient to apply because it does not require additional iterations or complex approaches that burden computation.

Interestingly, even with only one type of average (contra harmonic), this method has provided a significant performance improvement. This opens up a great opportunity for further exploration of other types of averages that have not been widely used in the development of numerical methods, such as the geometric mean, harmonic mean, and centroidal mean. Each has unique mathematical characteristics, thus potentially providing a variety of performance in different differential problem contexts.

This research provides important implications for the development of numerical methods, especially in solving ordinary differential equations [37]-[39]. Modification based on contra harmonic average can be adopted as a new approach in academic and practical fields, especially for systems that require high accuracy in simulation models such as in applied physics, system dynamics, to medical and environmental modeling [40], [41]. In addition, this method also shows potential to be applied to other numerical methods such as the Adams-Bashforth method or the predictor-corrector method.

This research is still limited to theoretical development and numerical simulation using contra harmonic average without systematic comparison to other types of averages. In addition, the scope of application is also still limited to basic case examples in the first-order ODEs system. Formal analysis of the stability of the method has not been carried out, and there has been no testing on stiff systems or complex nonlinear systems.

For further research, it is suggested to explore the modification of RK4 using other averages such as harmonic mean, geometric mean, and centroidal mean to see to what extent the type of average affects the accuracy and stability of the method. In addition, it is also important to develop a theoretical stability analysis of the method, as well as to conduct tests on various types of cases, including nonlinear and stiff differential systems. Implementation in various programming languages and performance tests on computational time can also be relevant development directions to support the practical application of this method.

4. CONCLUSION

The 4th Order Runge-Kutta method has the following general form:

$$y_{i+1} = y_i + \frac{h}{8}(k_1 + 3k_2 + 3k_3 + k_4)$$

After modifications were made using the Counterharmonic Average, a new form was obtained, namely:

$$y_{i+1} = y_i + \frac{h}{4}\left(\frac{k_1^2+k_2^2}{k_1+k_2} + 2\frac{k_2^2+k_3^2}{k_2+k_3} + \frac{k_3^2+k_4^2}{k_3+k_4}\right)$$

with

$k_1 = hf(y_i)$

$k_2 = hf\left(y_i + \frac{k_1}{3}\right)$

$k_3 = hf\left(y_i + k_1\left(\frac{5}{18} - \frac{\sqrt{73}}{18}\right) + k_2\left(\frac{7}{18} - \frac{\sqrt{73}}{18}\right)\right)$

$k_4 = hf\left(y_i + k_1\left(\frac{-5}{3} + \frac{\sqrt{73}}{3}\right) + k_2\left(\frac{19}{6} - \frac{\sqrt{73}}{2}\right) + k_3\left(\frac{-1}{2} - \frac{\sqrt{73}}{6}\right)\right)$

and the truncation error is

$$galat_{RKKCM} = h^5\left(\frac{17}{243}f^6 f_y f_{yyy} + \frac{11}{162}f^6 f_{yy}^2 + \left(\frac{1}{3} - \frac{2\sqrt{73}}{81}\right)f^5 f_y^2 f_{yy} + \left(\frac{73}{243} - \frac{5\sqrt{73}}{243}\right)f^4 f_y^4\right)$$

Based on numerical simulations using the RKKCM method, it is known that the results of this method have better accuracy compared to the RK-4 Kutta method before modification. In this study only using the Counter Harmonic Average to modify the RK-4 Kutta Method. Therefore, the researcher suggests that readers can further find new formulas by using other averages such as (Harmonic, Centroidal, and Geometric).

ACKNOWLEDGEMENTS

The author would like to express sincere gratitude to all parties involved in the completion of this research. Special thanks are extended to colleagues, mentors, and institutions who provided valuable insights, guidance, and support throughout the research process. Their contributions were instrumental in the successful development of this study.

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