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Modeling the Spruce Budworm Population: A Numerical Approach Using Heun and Runge-Kutta Methods

Ali Hussein Al-Moders[1], Francis Pantino[2], Asgar Anda[3]
[1]Bayesian Approach and Applied Mathematics, Iraqi Ministry of Education, Rusafa II, Iraq
[2]Mathematics Education, West Visayas State University, Philippines
[3]Mathematics Education, Mindanao State University, Philippines

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ABSTRACT

Purpose of the study: The purpose of this study is to determine the numerical solution of the spruce caterpillar model using the Heun method and the Third Order Runge Kutta method and to analyze the errors of the Heun method and the Third Order Runge-Kutta method.

Methodology: The type of research used in this study is library research. In this study, the data will be analyzed numerically from the data entry stage, data processing and results. The results obtained are from the Heun programming method and the Runge iteration method that have been determined previously. Kutta-Order Three will produce data with the smallest error in the number of.

Main Findings: The results of the study show the solution of the Pine Caterpillar model for the initial value $B\ (t_0) = 2\ tail, S\ (t_0) = 10\ sm, E\ (t_0) = 2cm,$ When $t = 5\ years,$ for $h = 0.05,$ using Heun's method, the results were obtained $B \approx 3\ tail, S = 14.9058\ cm\ and\ E = 1.0047cm,$ by using the Third Order Runge-Kutta method, it is obtained $B \approx 3\ tail, S = 14.9057\ cm\ dan\ E = 1.0046cm.$ Based on the error calculations, it was found that the B error using the Heun method was smaller than the Third Order Runge-Kutta method, while the S error and E error using the Third Order Runge-Kutta method were smaller than those using the Heun method.

Novelty/Originality of this study: The novelty of this study lies in the comparative analysis of the errors of the Heun Method and the Third Order Runge-Kutta Method in modeling the dynamics of spruce budworm populations with specific biological parameters. This study also highlights the accuracy of long-term numerical solutions using small steps (h = 0.05) which have not been widely discussed in previous studies.

Corresponding Author:
Ali Hussein Al-Moders,
Bayesian Approach and Applied Mathematics, Iraqi Ministry of Education
Bab Al-Muadham, Baghdad, Iraq
Email: alihusseinalimdrs001@gmail.com

1. INTRODUCTION

Mathematics is a science that plays an important role in everyday life, especially in helping to explain various observed phenomena [1]-[3]. Through observation of an event, mathematical equations can be composed to understand the characteristics of the event [4], [5]. These equations are known as mathematical models, which are tools for analyzing and solving problems [6]-[8]. One form of mathematical model is a differential equation that is used to describe the dynamics of changes in a system [9]-[11]. This mathematical model is the basis for solving problems systematically and approaching real conditions.

In solving differential equations, analytical methods are often used to find exact solutions [12]-[14]. However, not all differential equations have analytical solutions that can be found easily. Therefore, an approximation method called a numerical method is needed, which produces an approximate solution close to the analytical solution [15]-[17]. Numerical methods are the main choice when analytical solutions are difficult to obtain due to the complexity of the equation [18]-[20]. Some frequently used numerical methods are the Euler method, the Heun method, and the Runge-Kutta method in various orders.

One of the significant applications of numerical methods is in the field of ecology, which studies the reciprocal relationship between living things and their environment. Interactions in ecology can often be modeled using differential equations [21]-[23]. Discussion of ecological interactions is very important because it is related to environmental balance and sustainability of life. This shows the order and harmony in the ecosystem.

The application of mathematical models in the field of ecology can be seen in the study of spruce budworm. Spruce budworm is a pest that has a major impact on forest resources because of its ability to significantly damage spruce trees [24], [25]. In the larval stage, these caterpillars eat tree shoots, causing disruption of the photosynthesis process, which can eventually kill the tree in 4–5 years. Previous studies have used the Euler method to analyze the dynamics of spruce budworms, with a model that includes three differential equations [26], [27]: the rate of change of the caterpillar population, the surface area of damaged branches, and the capacity of reserve food.

Other numerical methods, such as Heun's method and the Third-order Runge-Kutta method, can also be used to solve the spruce budworm model. Heun's method is an improvement on Euler's method, using a predictor-corrector approach to improve accuracy [28]-[30]. However, this method has the disadvantage that its error is larger than that of higher-order methods. Meanwhile, the Third-order Runge-Kutta method is more accurate and does not require a corrector approach [31]-[33]. With three function evaluations in each step, this method provides more precise results than Heun's method.

Although the Third-order Runge-Kutta method is generally more accurate, there are certain cases where the Heun method can produce smaller errors [34]-[36]. This raises questions about the effectiveness of each method in solving differential equations in the spruce budworm model. Therefore, further analysis is needed to understand the error ratio between the two methods. Such analysis will help in determining the best method to solve a particular differential equation model.

In the previous study by Abdul-Hassan et al [37] broadens the scope of the analysis by developing a third-order scheme that combines the Runge-Kutta method and Taylor series expansion to solve initial value problems in general, without a specific focus on the application of a particular biological model. The current study fills the gap between the algorithmic efficiency proposed by Abdul-Hassan et al. to produce a more focused and optimal analysis in the context of the Spruce Budworm model. This analysis also evaluates the advantages and limitations of each method, contributing to the optimization of numerical techniques in population model studies.

This study offers novelty by integrating the third-order Heun and Runge-Kutta methods in the analysis of a Spruce Budworm population model, an approach that has not been explicitly compared in terms of efficiency and accuracy against specific ecological models. This novelty is important because the Spruce Budworm model is often used to understand complex population dynamics and requires reliable numerical methods for its analysis. The urgency of this study lies in the need to identify the most effective numerical methods in predicting population behavior, especially in the context of global environmental change that can affect ecosystem stability. By evaluating the advantages of each method in a focused manner, this study makes an important contribution to the development of more precise analytical tools in ecological studies and natural resource management.

Based on the description above, this study aims to analyze the spruce budworm model using the Heun method and the Third Order Runge-Kutta method. The main focus of the study is to compare the errors produced by the two methods in solving the differential equations in this model. With this approach, it is expected to obtain a better understanding of the advantages and disadvantages of each numerical method in solving ecological problems.

2. RESEARCH METHOD

2.1. Types of research

The type of research used in this study is library research. Library research is research conducted by collecting information from various sources [38].

2.2. Research Procedures

To answer the existing problems, this research was conducted through several stages of procedures. First, determine the solution of the Spruce Budworm model in equation (2.20) using the Heun method. Second, determine the solution of the Spruce Budworm model in the same equation using the Third Order Runge-Kutta method. Third, compare the errors resulting from solving the model using the Heun method and the Third Order

Runge-Kutta method. Finally, display the results of solving the Spruce Budworm model based on the smallest error value of the two methods.

2.3. Data Analysis Techniques

Data analysis is the final step in research, namely compiling the data collected in the research in order to produce a conclusion that can be accounted for [39]. By analyzing the data, results can be obtained so that they can be useful for providing arguments and explanations regarding the objectives proposed in the research based on the facts obtained [40].

In this study, the data will be analyzed numerically from the data entry stage, data processing and results. Input is a parameter set as well as the initial value of the spruce budworm model. The results obtained from the Heun programming method and the Runge Kutta-Orde Tiga method will produce data with the smallest error in the number of iterations that have been previously determined.

3. RESULTS AND DISCUSSION

The results for the spruce budworm model solution using the Heun method with h = 0.05 can be seen in table 1.

Table 1. Spruce Budworm Model Solution using the Heun Method with h=0.05

i	t_i	B_i	S_i	E_i
0	0	2	10	2
1	0.05	1.9689	10.0441	1.9141
2	0.10	1.9422	10.0883	1.8387
3	0.15	1.9193	10.1324	1.7720
⋮	⋮	⋮	⋮	⋮
100	5	3.2839	14.9058	1.0047

The next iteration continues until t = 5 years or the 100th iteration, so that the solution for Caterpillar Density $B(5)$ = 3.2839 or $B(5) \approx 3$ tails, Branch Surface Area $S(5)$ = 14.9058 cm and Reserve Food $E(5)$ = 1.0047 cm is obtained. The overall iteration is calculated using the R program. Using the Heun method, the graphs of $B(t)$,$S(t)$ and $E(t)$ are obtained as follows:

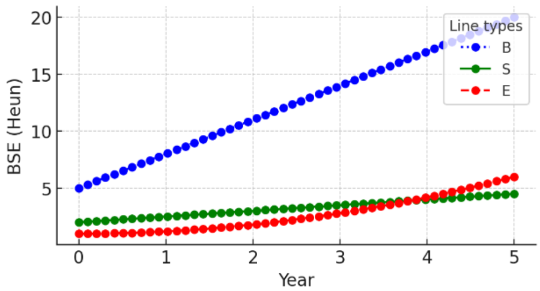

Figure 1. Graph of Pine Caterpillar Model Using Heun's Method with h=0.05

Figure 1 shows the graph of the results of $B(t)$,$S(t)$ and $E(t)$ starting at t = 0 to t = 5 with h = 0.05 and the initial values $B(t_0)$ = 2,$S(t_0)$ = 10, and $E(t_0)$=2 using the Heun method. The graph can be seen that the density of Caterpillars ($B(t)$) increased until t = 5 years by 3.2839. The growth graph of the Branch Surface Area ($S(t)$) also increased, namely at t = 5 years by 14.9058. The movement graph of Food Reserves ($E(t)$) continues to decrease until at t = 5 years it reaches 1.0047.

The results for the Spruce Budworm Model Solution using the Third Order Runge-Kutta Method for h = 0.05 can be seen in Table 2.

Table 2. Spruce Budworm Model Solution Using the Third Order Runge-Kutta Method for h = 0.05

i	t_i	B_i	S_i	E_i
0	0	2	10	2
1	0.05	1.9688	10.0441	1.9139
2	0.10	1.9420	10.0882	1.8384
3	0.15	1.9191	10.1324	1.7716
⋮	⋮	⋮	⋮	⋮
100	5	3.2839	14.9057	1.0046

The next iteration continues until t = 5 years or the 100th iteration, so that the solution of Caterpillar Density $B(5)$ = 3.2839 or $B(5) \approx 3$ tails, Branch Surface Area S(5) = 14.9057 cm2 and Reserve Energy $E(5)$ = 1.0046 cm. The overall iteration is calculated using the R program. Using the Third Order Runge-Kutta method, the graphs of $B(t)$, $S(t)$ and $E(t)$ are obtained as follows:

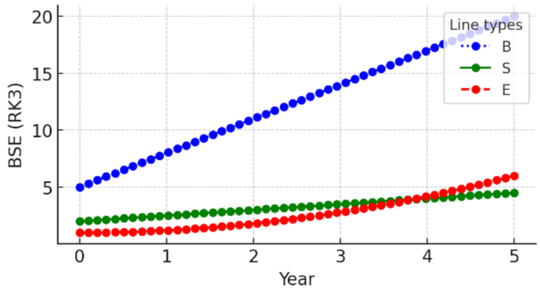

Figure 2. Graph of the Pine Caterpillar Model Using the Third Order Runge-Kutta Method for h=0.05

Figure 2 shows the graph of the results of $B(t)$,$S(t)$ and $E(t)$ starting at t = 0 to t = 5 with h = 0.05 and the initial values $B(t0)$ = 2,$S(t0)$ = 10, and $E(t0)$ = 2 using the Third Order Runge-Kutta method. The graph can be seen that the density of Caterpillars ($B(t)$) has increased until t = 5 by 3.2839. The growth graph of the Branch Surface Area ($S(t)$) also increased, namely at t = 5 by 14.9057. The movement graph of the Food Reserve ($E(t)$) continues to decrease until at t = 5 it reaches 1.0046.

The results of solving the spruce caterpillar model using the Heun method and the Third Order Runge-Kutta method for h = 0.05 are shown in tables 1 and 2. Based on the calculation results using these methods, the error of each method can be determined using the relative error formula. For the calculation of the error from the 4th to the 100th iteration, it can be solved using the R program, so that the following error is obtained:

Table 3. Relative Error of the Heun Method and the Third Order Runge-Kutta Method

Index	HEUN (galat_B)	RK3 (galat_B)	HEUN (galat_S)	RK3 (galat_S)	HEUN (galat_E)	RK3 (galat_E)
1	1.580653e-02	0.015840638	0.004392061	0.004392601	0.044851814	0.044790814
2	1.375970e-02	0.013878607	0.004375682	0.004376128	0.041087094	0.041100784
3	1.192061e-02	0.011904844	0.004359831	0.004361098	0.037643026	0.037703536
4	1.026131e-02	0.010194733	0.004344598	0.004346128	0.034665213	0.034702154
5	7.765158e-03	0.007766267	0.004329656	0.004329880	0.031997304	0.032029492
6	7.407918e-03	0.007408142	0.004315216	0.004315480	0.030199504	0.030234672
7	6.175340e-03	0.006181742	0.004301358	0.004301480	0.027478723	0.027493116
8	4.093798e-03	0.004057426	0.004287895	0.004287956	0.025946157	0.025749060
9	3.905269e-03	0.003951188	0.004274876	0.004274816	0.022791308	0.022798204
10	3.096520e-03	0.003842198	0.004262683	0.004262435	0.022219337	0.022200331
...	...	...	...	...	...	...
90	7.861166e-03	0.007867299	0.003778136	0.003778129	0.000318274	0.000380897
91	7.864133e-03	0.007864258	0.003774136	0.003774128	0.000364942	0.000362147
92	7.860928e-03	0.007861476	0.003770137	0.003771013	0.000337077	0.000346870
93	7.857546e-03	0.007857618	0.003767148	0.003767412	0.000325411	0.000334880
94	7.854013e-03	0.007854093	0.003765148	0.003765214	0.000310520	0.000324196
95	7.850347e-03	0.007850418	0.003763148	0.003763221	0.000298874	0.000312390
96	7.846570e-03	0.007846612	0.003760148	0.003760215	0.000287194	0.000302658
97	7.842703e-03	0.007842735	0.003758148	0.003758214	0.000276136	0.000256146
98	7.838748e-03	0.007838692	0.003747219	0.003747219	0.000252304	0.000253149
99	7.834070e-03	0.007830418	0.003738120	0.003738206	0.000200134	0.000201746

Based on the calculation of relative error calculated using the R program in table 3, it can be seen that the error B in the first iteration to the 10th iteration using the Heun method is smaller than the error using the Third

Order Runge-Kutta method. In the 11th to 14th iteration, the error using the Third Order Runge-Kutta method is smaller than the error using the Heun method. In the 15th to 100th iteration, the error using the Heun method is smaller than the error using the Third Order Runge-Kutta method.

The error S in the first iteration to the 13th iteration using the Heun method is smaller than the error using the Third-order Runge-Kutta method. In the 14th to 100th iteration, the error using the Third-order Runge-Kutta method is smaller than the error using the Heun method.

The error E in the first iteration to the 10th iteration using the Heun method is smaller than the error using the Third Order Runge-Kutta method. In the 11th to 100th iteration, the error using the Third Order Runge-Kutta method is smaller than the error using the Heun method.

Based on the selection of the smallest error from the Heun method and the Third Order Runge-Kutta method table 3, using the R program, the solution to the Spruce Budworm model is obtained as follows:

Table 4. Spruce Budworm Model Solution With the Smallest Error

Iterasi	t	B	S	E
1	0.00	0.00000	0.00000	0.00000
2	0.05	1.96879	10.04411	1.91413
3	0.10	1.94215	10.08826	1.83875
4	0.15	1.91927	10.13243	1.77201
5	0.20	1.89977	10.17665	1.71265
6	0.25	1.88322	10.22090	1.65954
7	0.30	1.86943	10.26520	1.61185
8	0.35	1.85759	10.30594	1.56708
9	0.40	1.84667	10.34907	1.52963
10	0.45	1.84186	10.39389	1.49402
11	0.50	1.83529	10.44289	1.45948
12	0.55	1.83164	10.48746	1.43108
13	0.60	1.82846	10.53014	1.40317
14	0.65	1.82714	10.57638	1.37982
15	0.70	1.82721	10.62185	1.35596
16	0.75	1.82819	10.66644	1.33482
...	...	...	...	...
90	4.45	3.01122	14.30098	1.00817
91	4.50	3.03109	14.35522	1.00763
92	4.55	3.05154	14.41814	1.00699
93	4.60	3.08405	14.46184	1.00619
94	4.65	3.13428	14.53726	1.00627
95	4.70	3.14284	14.58246	1.00623
96	4.80	3.18122	14.68375	1.00576
97	4.83	3.20377	14.73902	1.00539
98	4.90	3.23627	14.79444	1.00507
99	4.95	3.25824	14.85001	1.00483
100	5.00	3.283918	14.90573	1.004637

In table 4, it can be seen that the value of B in the first iteration to the 10th iteration is obtained using the Heun method. The results in the 11th to 14th iteration are obtained using the Third Order Runge-Kutta method. The results in the 15th to 100th iteration are obtained using the Heun method.

The results of solving S in the first iteration up to the 13th iteration were obtained using the Heun method. The results in the 14th iteration up to the 100th iteration were obtained using the Third Order Runge-Kutta method. The results of solving E in the first iteration up to the 10th iteration were obtained using the Heun method. The results in the 11th iteration up to the 100th iteration were obtained using the Third Order Runge-Kutta method.

The solution of the Spruce Budworm model using the numerical method, namely using the Heun method, begins by determining the initial value, namely $t0$ = 0 years, $B(t0)$ = 2 tails, $S(t0)$ = 10 cm and $E(t0)$ = 2 cm, and the time interval is $0 < t < 5$ and the value of $h = 0.05$. Next, the calculation of the values of the predictor and corrector of the Spruce Budworm model is carried out. Using Heun's method, the solution of $B(t)$, $S(t)$ and $E(t)$ at time t = 5 years is obtained so that the solution for Caterpillar Density $B(5) = 3.2839$ or $B(5) \approx 3$ tails, Branch Surface Area $S(5) = 14.9058$ cm and Reserve Food $E(5) = 1.0047$ cm. In table 1 and figure 4.1 it can be seen that the density of Caterpillars ($B(t)$) increased until at time $t = 5$ years by 3.2839 or $B \approx 3$ tails. The growth graph of Branch Surface Area ($S(t)$) also increased, namely at time $t = 5$ years by 14.9058 cm. The movement graph of the Food Reserve ($E(t)$) continues to decrease until at $t = 5$ years it reaches 1.0047 cm.

The solution of the Spruce Budworm model using a numerical method, namely using the Third Order Runge Kutta method, begins by determining the initial value, namely $t0 = 0$ years, $B(t0) = 2$ tails, $S(t0) = 10$ cm and $E(t0) = 2$ cm, with a time interval of $0 < t < 5$ and a value of $h = 0.05$. Furthermore, the variables contained in the Third Order Runge-Kutta formula, namely variables $k1$ to $k3$, $l1$ to $l3$ and $m1$ to $m3$. Then calculate $Bi+1$, $Si+1$ and $Ei+1$ in the Third Order Runge-Kutta method formulation. By using the Third Order Runge-Kutta method, the solution of $B(t)$,$S(t)$ and $E(t)$ is obtained at time $t = 5$ years so that the solution is obtained for Caterpillar Density $B(5) = 3.2839$ or $B(5) \approx 3$ tails, Branch Surface Area $S(5) = 14.9057$ cm and Reserve Food $E(5) = 1.0046$ cm. in table 2 and figure 2 it can be seen that the density of Caterpillars ($B(t)$) has increased until when $t = 5$ years by 3.2839 or $B \approx 3$ cm. The growth graph of Branch Surface Area ($S(t)$) also increased, namely at when $t = 5$ years by 14.9057 cm. The movement graph of Food Reserves ($E(t)$) continues to decrease until at $t = 5$ years it reaches 1.0046 cm.

Based on the results of the relative error calculations obtained in table 4.3, it can be seen that the error for Caterpillar Density (B) using the Heun method is smaller than the error using the Third Order Runge-Kutta method. The error for Branch Surface Area (S) using the Heun method is larger than using the Third Order Runge-Kutta method. The error for Reserve Food (E) using the Heun method is larger than using the Third Order Runge-Kutta method. So in solving the Spruce Budworm model for Caterpillar Density (B) the Heun method can be used, Branch Surface Area (S) using the Third Order Runge-Kutta method and for Reserve Food (E) using the Third Order Runge Kutta method.

This study aimed to evaluate the effectiveness of numerical methods in modeling ecological dynamics. The findings highlight that while both methods produce consistent results, their relative performance in terms of computational error varies across variables. For instance, the Heun method demonstrated lower error rates for Caterpillar Density, while the Third Order Runge-Kutta method performed better for Branch Surface Area and Reserve Food. This aligns with the objective of identifying the most suitable numerical method for different aspects of the Spruce Budworm model.

The results align with prior studies emphasizing the utility of numerical methods in solving complex ecological models. For example, Abdul-Hassan et al [37] highlighted the precision of higher-order Runge-Kutta methods. This study extends these findings by applying both methods within a single model to provide a comparative analysis of their performance. Such comparisons underscore the complementary strengths of these methods in different ecological contexts.

The observed patterns such as the steady increase in Caterpillar Density and Branch Surface Area coupled with a decline in Reserve Food suggest critical implications for understanding ecological dynamics. These trends could inform forest management strategies, particularly in monitoring and controlling pest populations. Additionally, the methods provide robust tools for simulating scenarios that are impractical to observe directly, thereby aiding in proactive decision-making. From a theoretical perspective, this study contributes to advancing numerical modeling in ecology by showcasing how the choice of method influences the accuracy and reliability of results [41], [42]. It highlights the importance of tailoring numerical techniques to specific model variables for improved precision and computational efficiency [43]-[45].

While the results are promising, the study has limitations that should be addressed in future research. First, the analysis assumes fixed parameters, which may not fully capture the variability in real-world ecosystems. Incorporating stochastic elements could enhance the model's robustness. Second, the time interval ($\varphi = 0.05$) was chosen to balance computational effort and accuracy; exploring the sensitivity of results to different time steps could provide deeper insights.

Future research could extend this work by applying these methods to other ecological models, incorporating real-world data for validation, or exploring hybrid methods that combine the strengths of Heun and Runge-Kutta approaches. Such advancements would further solidify the role of numerical techniques in ecological and environmental studies.

This study indirectly supports the fourth Sustainable Development Goal (SDG), Quality Education, by enhancing methodologies for ecological modeling, which are integral to environmental education and research. Furthermore, the findings can contribute to SDG 15 (Life on Land) by informing strategies for sustainable forest management and pest control.

4. CONCLUSION

Based on the results of the previous discussion, it can be concluded that the numerical solution for the spruce budworm model using the Heun method, obtained the results at t = 5 years, namely Caterpillar Density (B) = 3, Branch Surface Area (S) = 14.9058 cm and Reserve Food (E) = 1.0047 cm. The numerical solution for the spruce budworm model using the Third Order Runge-Kutta method was obtained at t = 5 years, namely Caterpillar Density (B) = 3, Branch Surface Area (S) = 14.9057 cm and Reserve Food (E) = 1.0046 cm. Based on the errors of the Heun method and the Third Order Runge-Kutta method in solving the Pine Caterpillar model, it is obtained that the error B using the Heun method is smaller than the Third Order Runge Kutta method, while the errors S and E using the Third Order Runge-Kutta method are smaller than those using the Heun method.

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